Pseudorandom Generators vs. Derandomization for Logspace Algorithms

(Paper title: "Targeted Pseudorandom Generators, Simulation Advice Generators, and Derandomizing Logspace")

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STOC

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  - Assume the following derandomization statement:

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- Best derandomization (Saks, Zhou '99):

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- Equivalence of PRGs and derandomization would itself give a derandomization!

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- Crucial bonus feature: PRG doesn't see "source code"!

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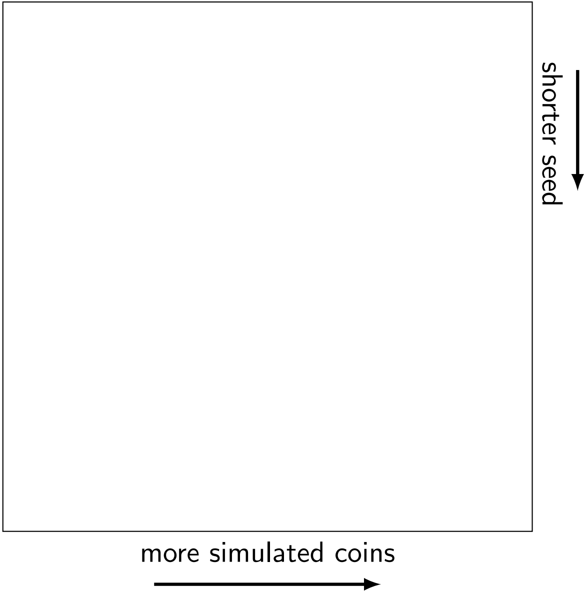
  - $\implies$ simulator with seed length/space complexity

$$O(\log^{3/2} n + \log^{3/2} n) = O(\log^{3/2} n)$$

Proof of main result



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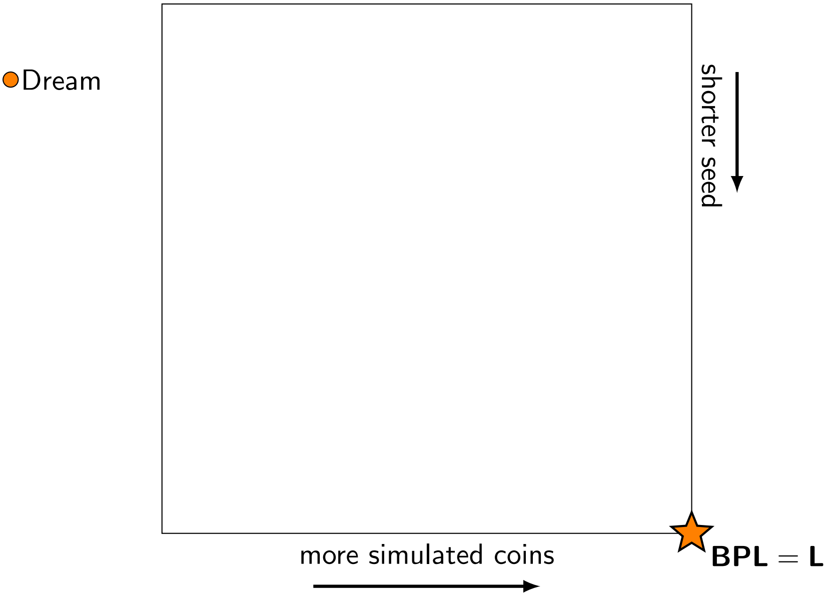

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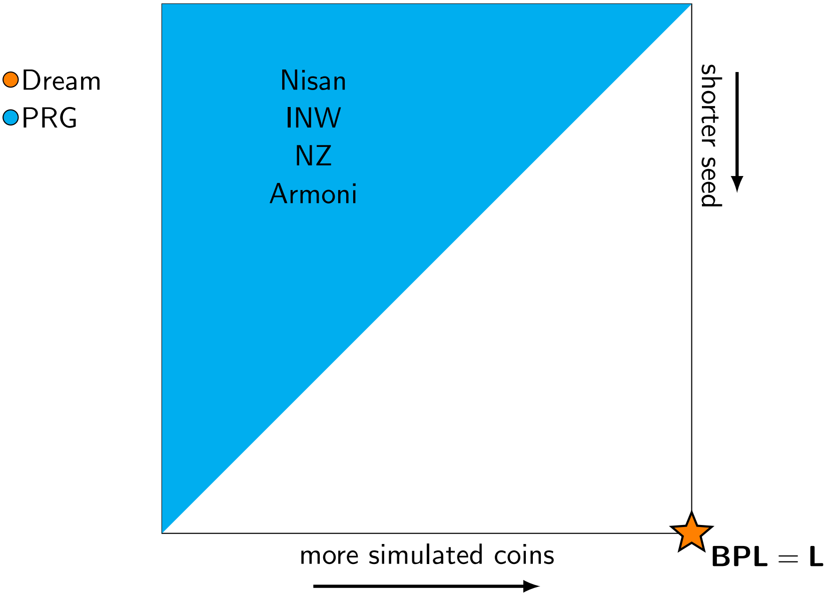

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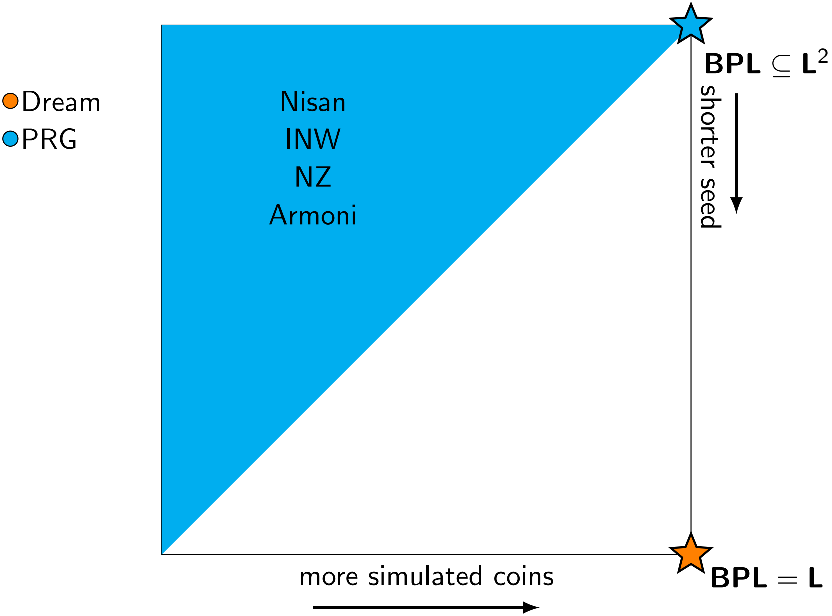

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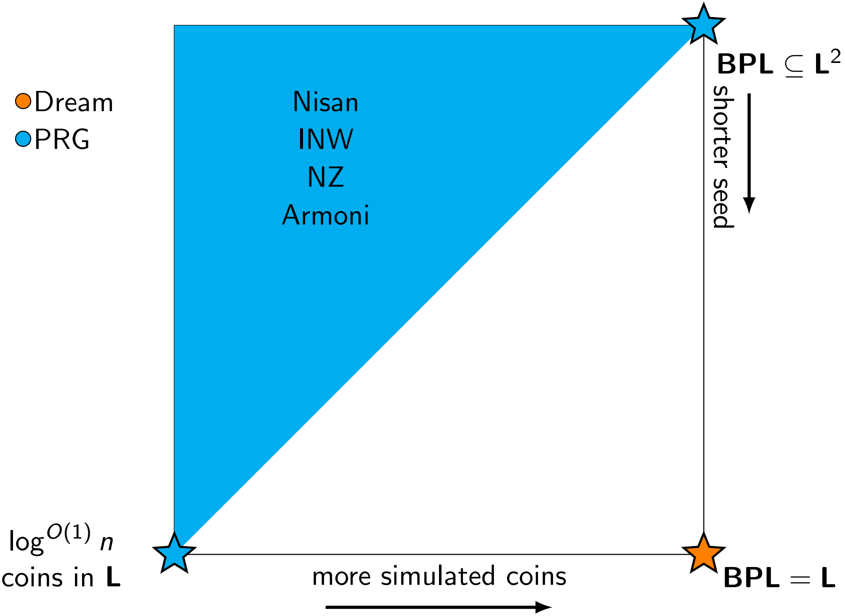

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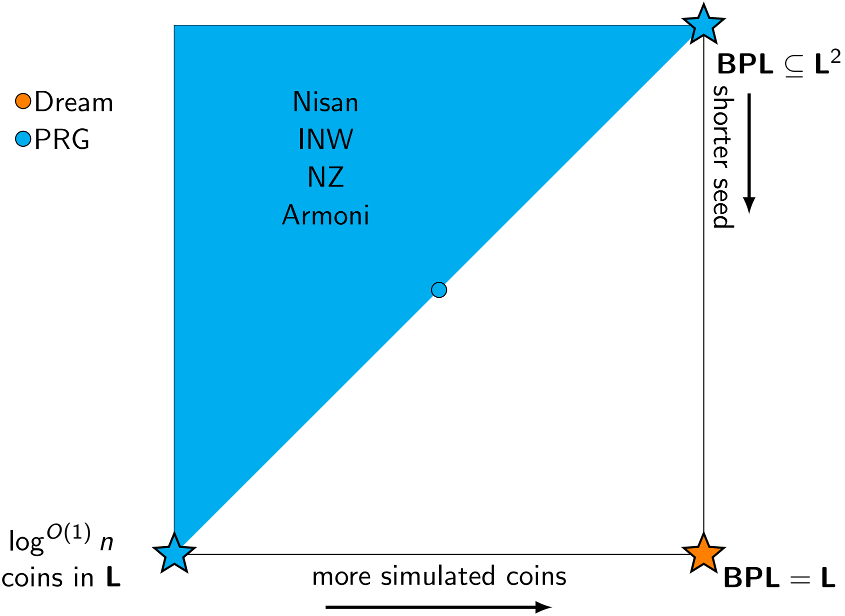

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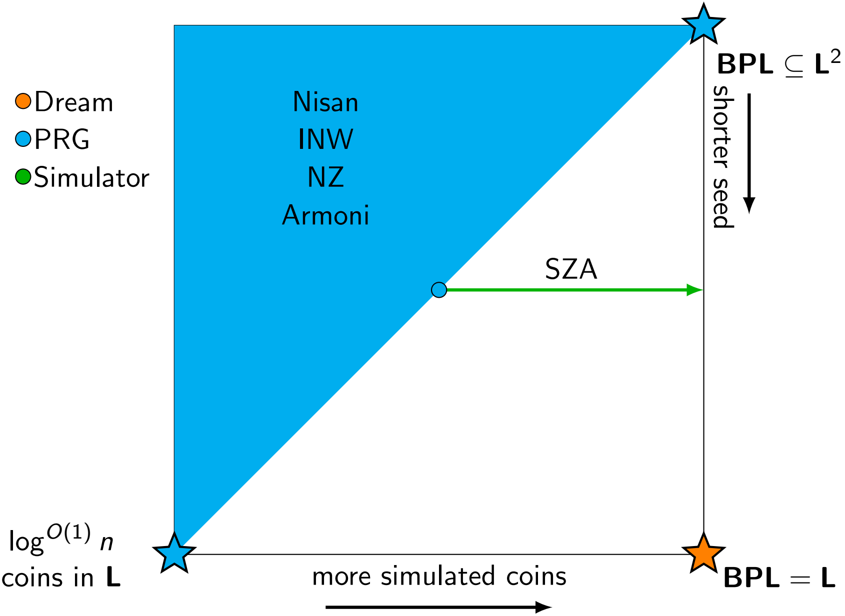

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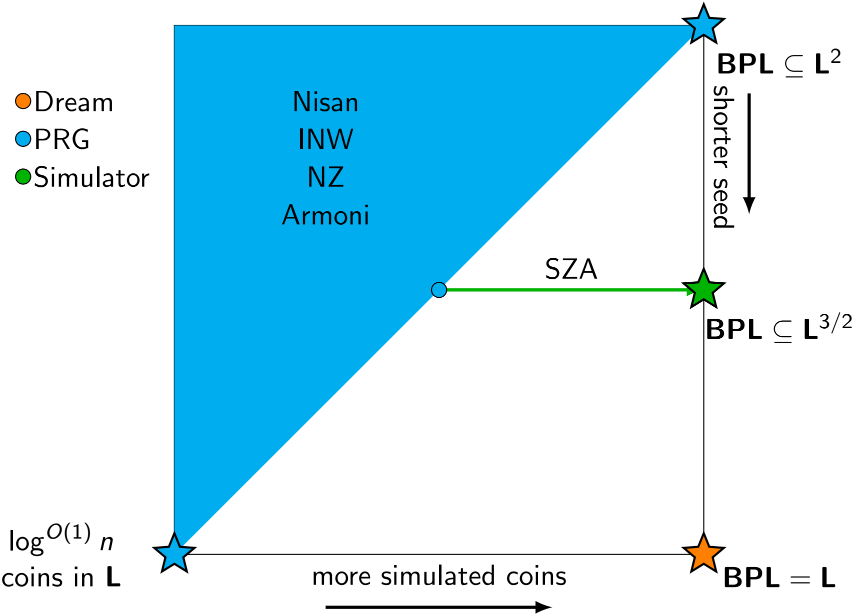

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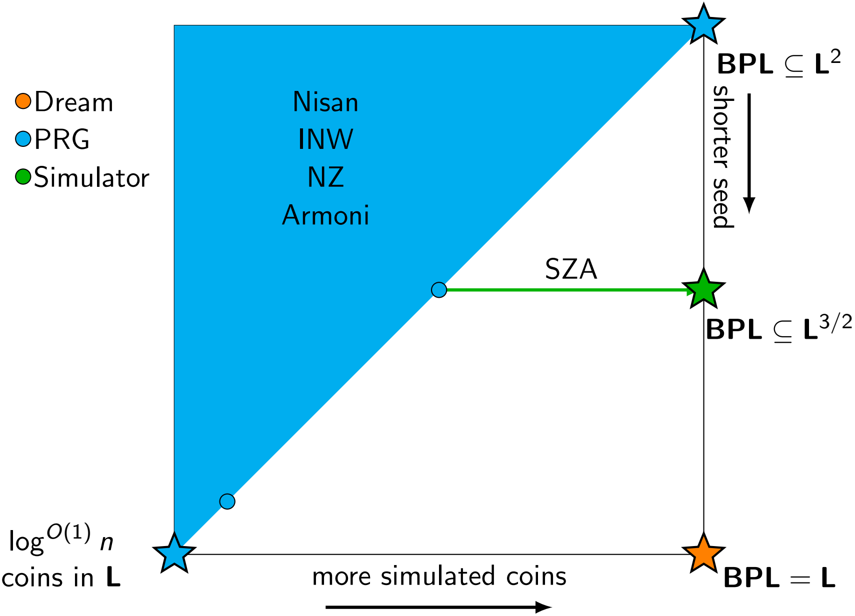

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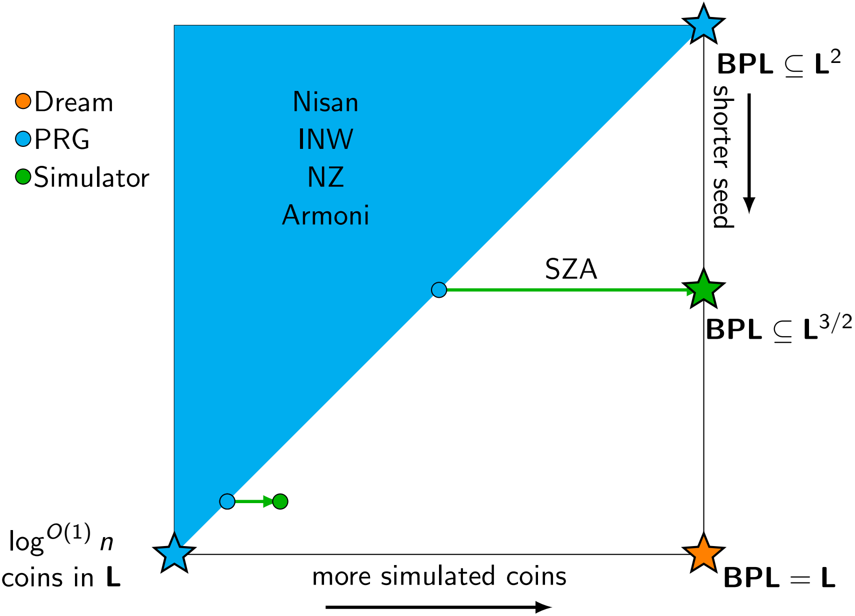

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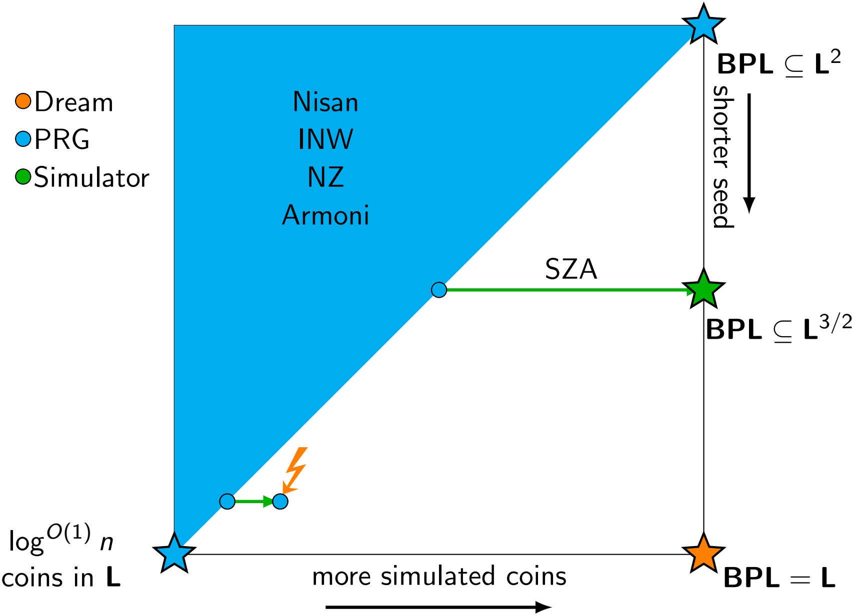

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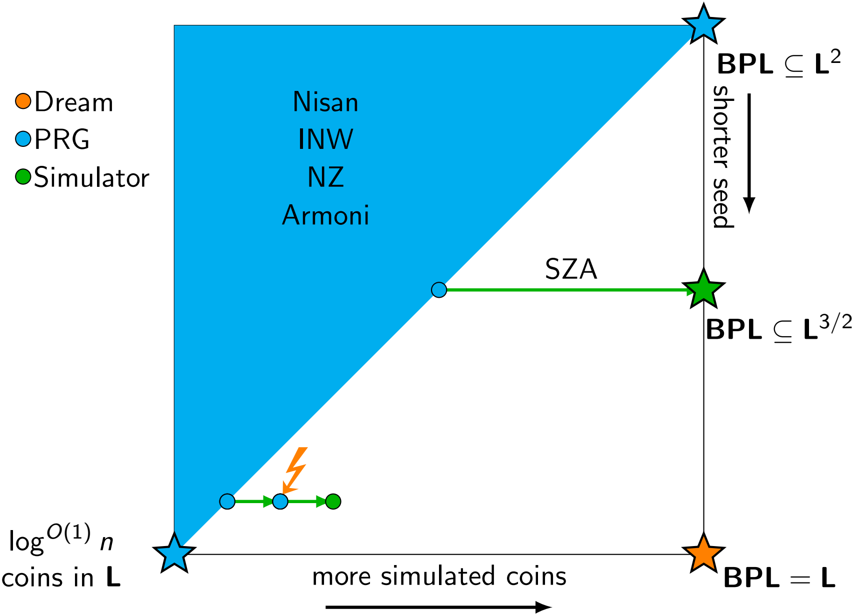

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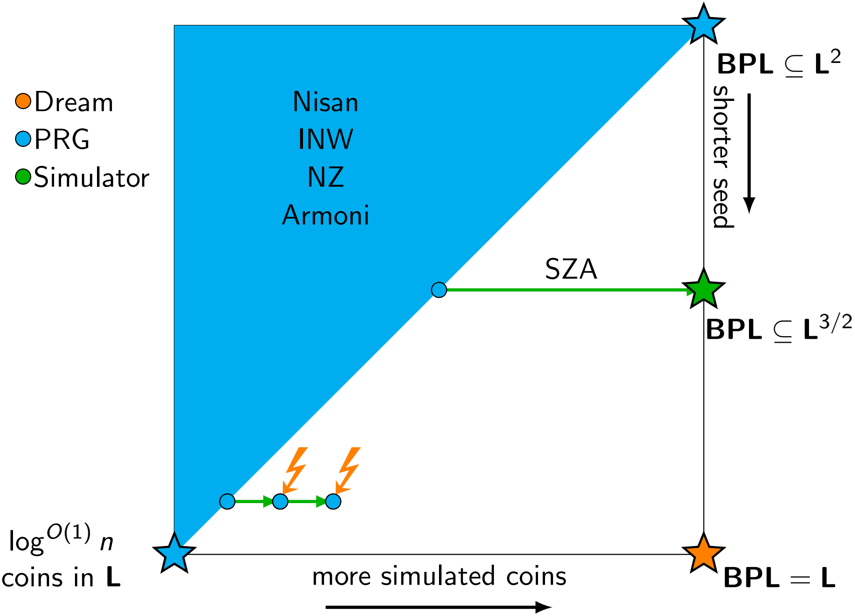

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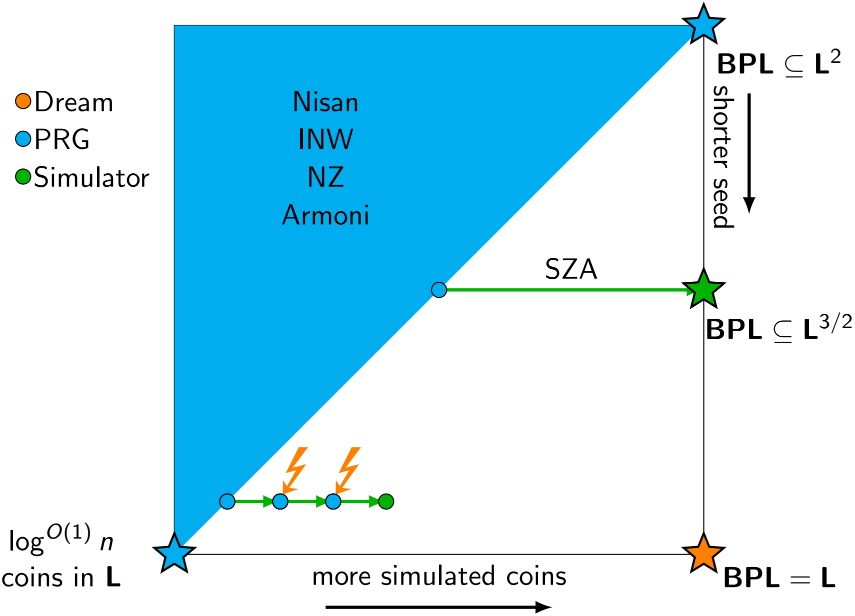

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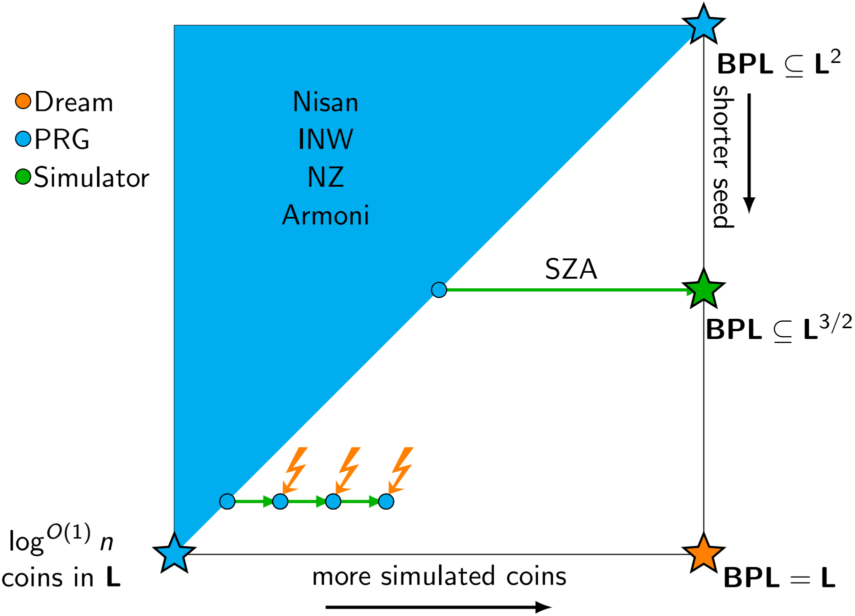

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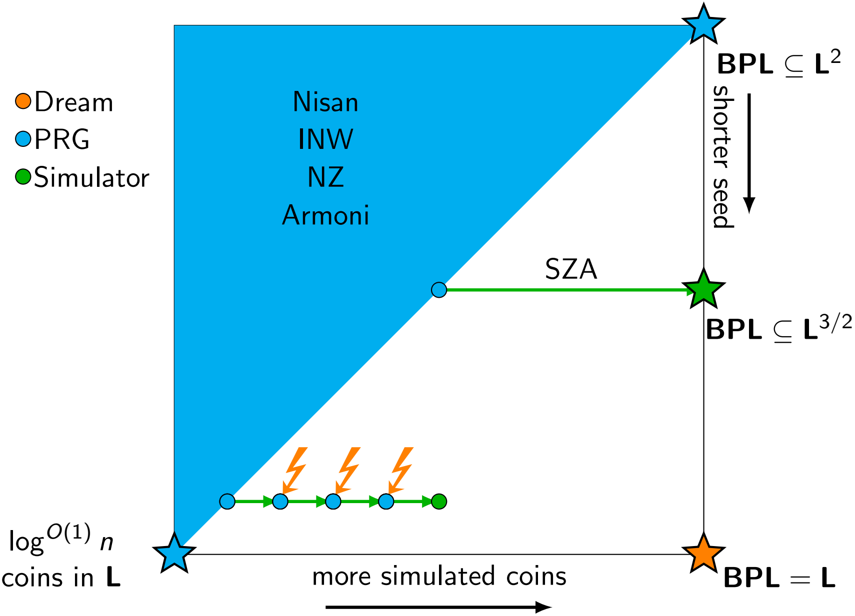

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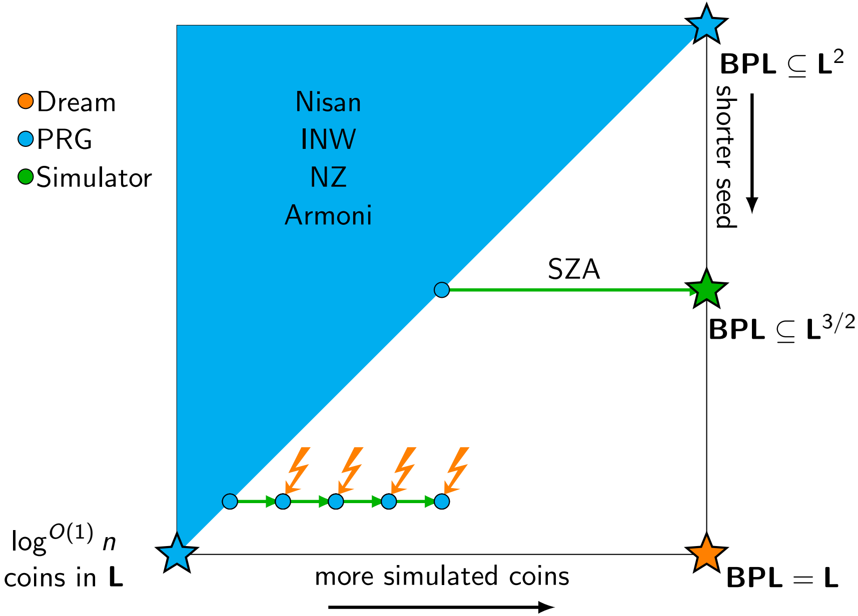

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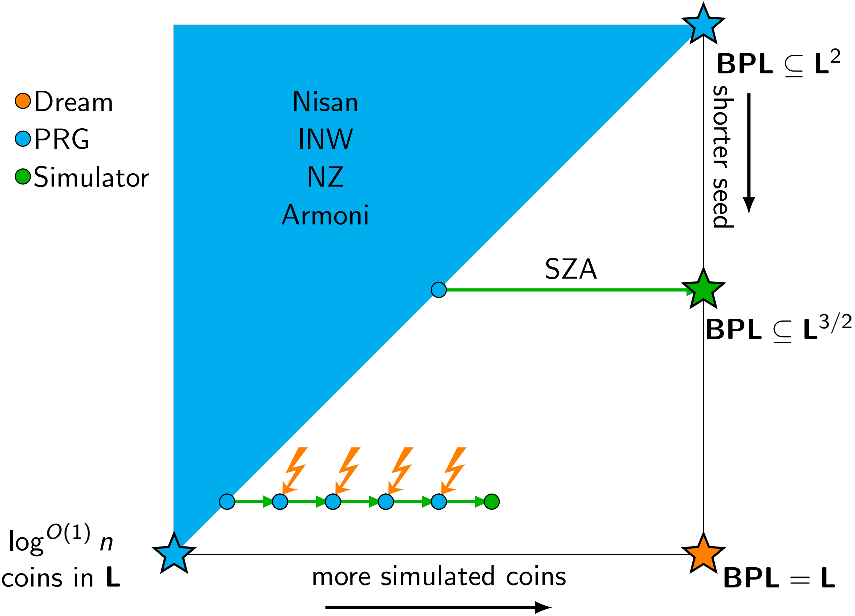

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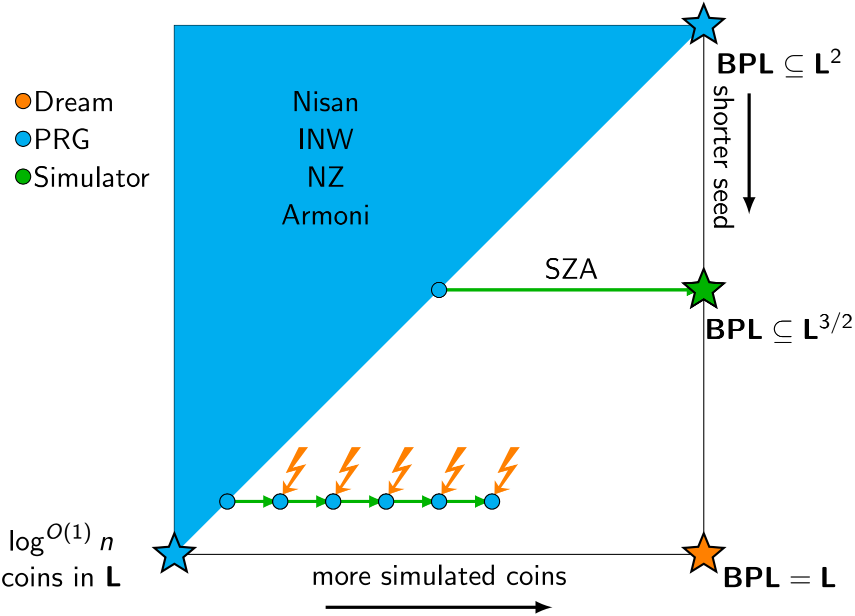

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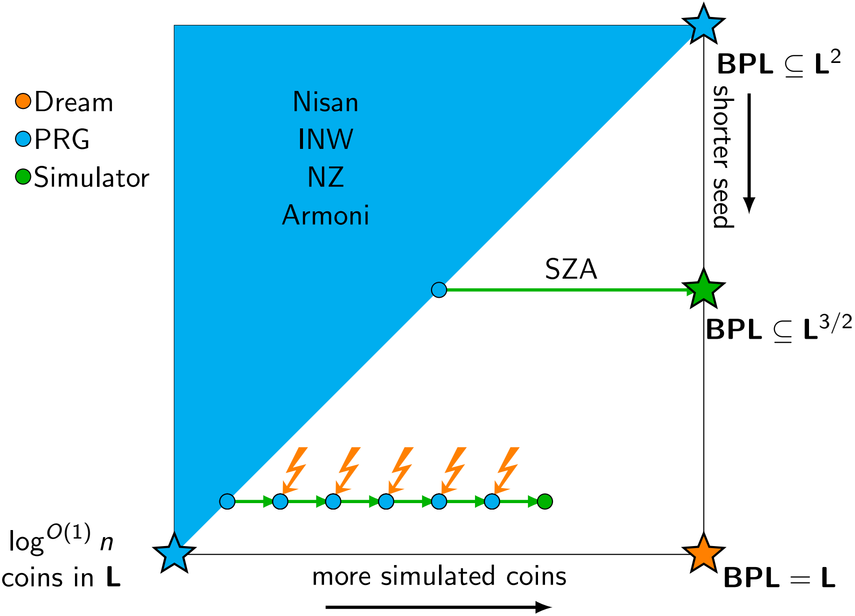

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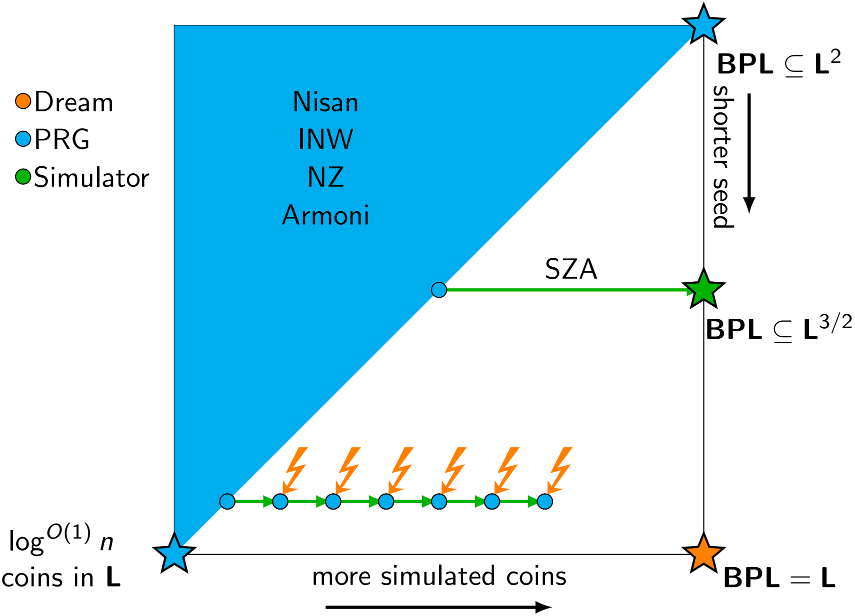

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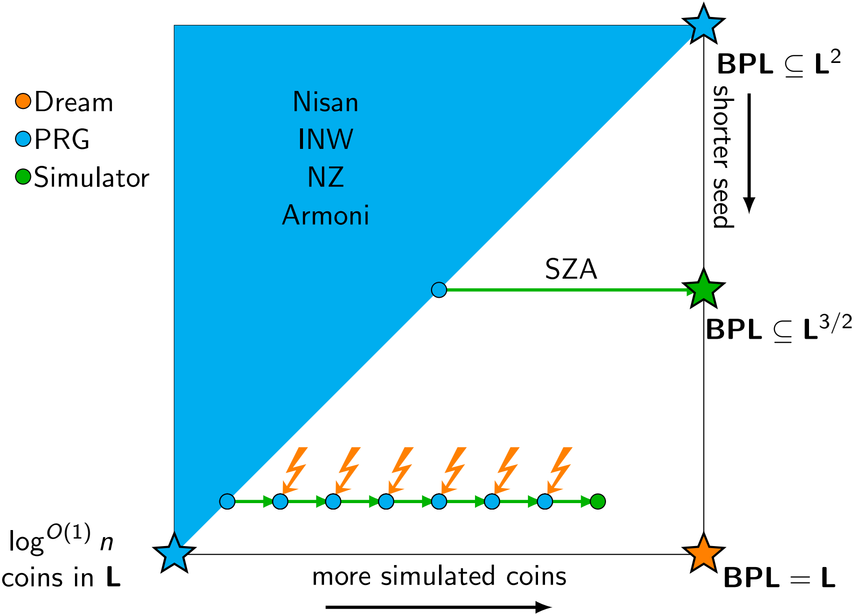

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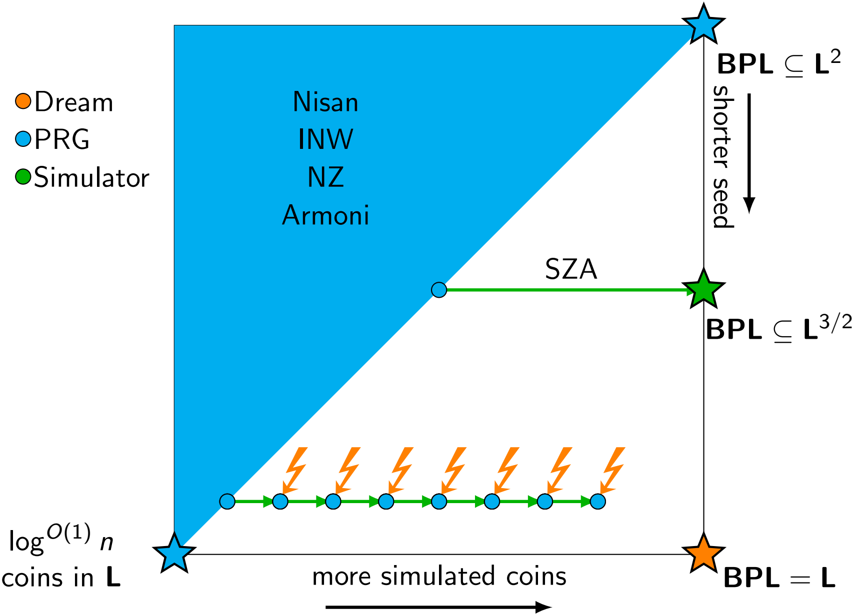

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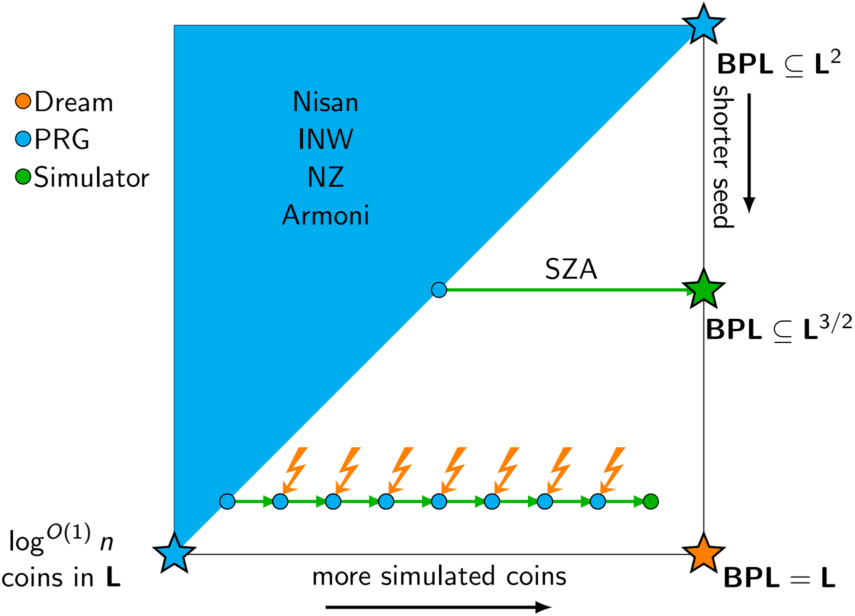

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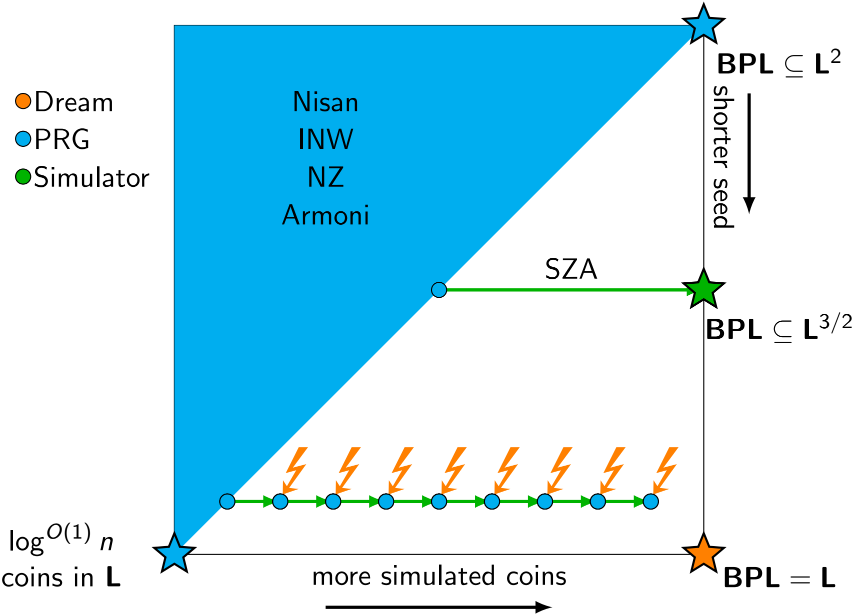

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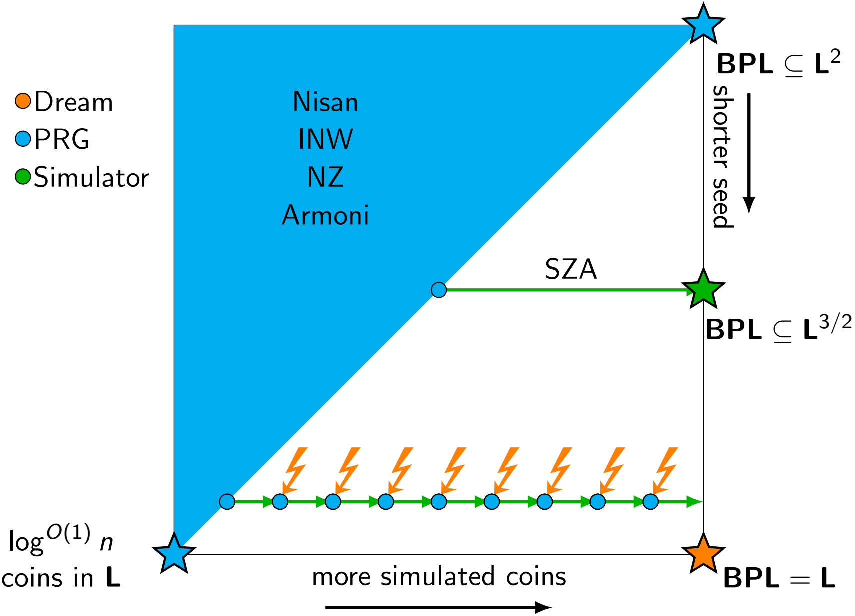

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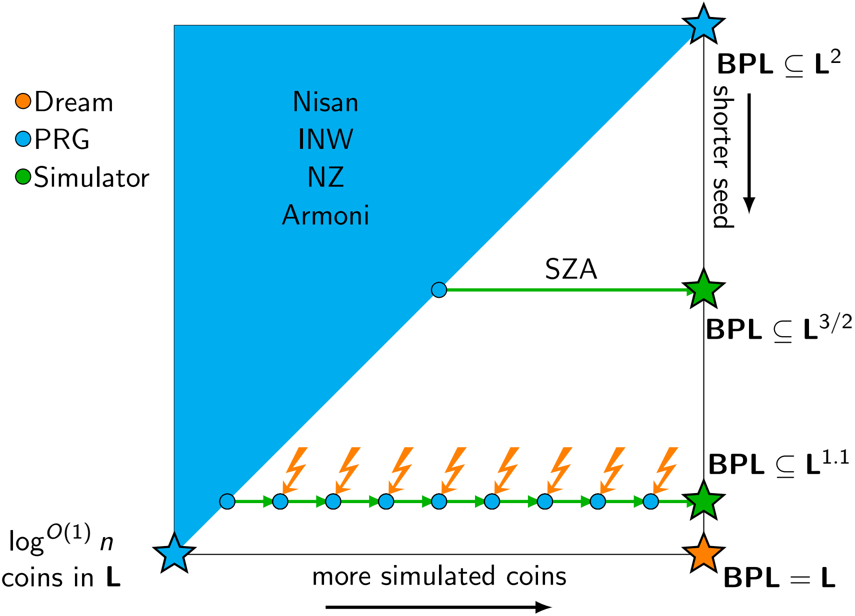
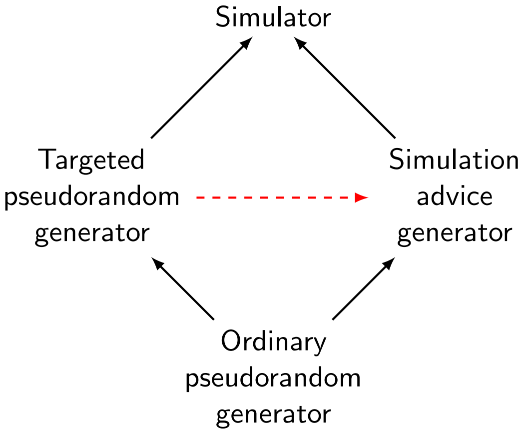


Stronger version of main result



Theorem: Dashed arrow transformation exists if and only if

$$\bigcap_{\alpha>0} \textbf{promise-BPSPACE}(\log^{1+\alpha} n) = \bigcap_{\alpha>0} \textbf{promise-DSPACE}(\log^{1+\alpha} n)$$

Conclusion

- This material is based upon work supported by
  - NSF GRFP Grant No. DGE-1610403
  - NSF Grant No. NSF CCF-1423544
- Thanks for your attention!
- Any questions?